Logarithmic Functions and Simple Interest

Finite Math

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Quiz

In the formula for exponential growth, $A = ce^{rt}$, if the growth rate were 5%, which letter would you replace with .05?

Inverse Functions

The inverse of a function is given by running the function backwards.
But when can we do this?
Consider the function $f(x) = x^2$. If we run f backwards on the value 1, what x-value do we get?
Since $(1)^2 = 1$ and $(-1)^2 = 1$, we get *two* values when we run x^2 backward! So x^2 is not invertible.

Inverse Functions

We know that not every function is invertible. In order for a function to be invertible, we need each range value to come from *exactly one* domain value. We call such functions *one-to-one*.

If we have a one-to-one function

$$y = f(x)$$

we can form the *inverse function* by switching x and y and solving for y:

$$x = f(y) \overset{\text{solve for } y}{\longrightarrow} y = f^{-1}(x).$$

Logarithms

We will focus on one particular inverse function: the inverse of the function $f(x) = b^x$ ($b > 0$, $b \neq 1$).

Definition (Logarithm)

The logarithm of base b is defined as the inverse of b^x. That is,

$$y = b^x \iff x = \log_b y.$$

Since the domain and range switch when we take inverses, we have

function	domain	range
$f(x) = b^x$	$(-\infty, \infty)$	$(0, \infty)$
$f(x) = \log_b x$	$(0, \infty)$	$(-\infty, \infty)$

Graphing a Logarithmic Function

Example

Sketch the graph of $f(x) = \log_2 x$.

Properties of Logarithms

Property (Properties of Logarithms)

Let $b, M, N > 0$, $b \neq 1$, and p, x be real numbers. Then

1. $\log_b 1 = 0$
2. $\log_b b = 1$
3. $\log_b b^x = x$
4. $b^{\log_b x} = x$
5. $\log_b MN = \log_b M + \log_b N$
6. $\log_b \frac{M}{N} = \log_b M - \log_b N$
7. $\log_b M^p = p \log_b M$
8. $\log_b M = \log_b N$ *if and only if* $M = N$

The Natural Logarithm

Just as with exponential functions, if we choose our base to be the number e, we get a special logarithm, the *natural logarithm*.

$$\log_e x = \ln x.$$

We can actually rewrite a logarithm in any base in terms of ln:

$$\log_b x = \frac{\ln x}{\ln b}$$

(See the textbook for a proof of this.)

Using Properties of Exponents and Logarithms

Example

Solve for x in the following equations:

(a) $7 = 2e^{0.2x}$

(b) $16 = 5^{3x}$

(c) $8000 = (x - 4)^3$

Reminder of Some Exponent Types

A quick reminder of different types of exponents:

- $a^{-n} = \frac{1}{a^n}$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$
  - $a^{1/2} = \sqrt{a}$
  - $a^{1/3} = \sqrt[3]{a}$
- $a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

Now You Try It!

Example

Solve for x in the following equations:

(a) $75 = 25e^{-x}$

(b) $42 = 7^{2x+3}$

(c) $200 = (2x - 1)^5$

Solution

(a) $x \approx -1.09861$

(b) $x \approx -0.53961$

(c) $x \approx 1.94270$

Applications

Recall that exponential growth/decay models are of the form

$$A = ce^{rt}.$$

Using the natural logarithm, we can solve for the rate of growth/decay, r, and the time elapsed, t. Let's see this in an example.

Example

The isotope carbon-14 has a half-life (the time it takes for the isotope to decay to half of its original mass) of 5730 years.

(a) *At what rate does carbon-14 decay?*

(b) *How long would it take for 90% of a chunk of carbon-14 to decay?*